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Approximation of Bit Error Rates in Digital Communications

Graham V. Weinberg and Sharon Lee

Electronic Warfare and Radar Division
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ABSTRACT

This report investigates the estimation of bit error rates in digital communications, motivated by recent work in [6]. In the latter, bounds are used to construct estimates for bit error rates in the case of differentially coherent quadrature phase-keying with Gray coding over an additive white Gaussian noise channel. By analysing Marcum's Q-Function, which is an integral part of bit error rate expressions, we derive more direct methods of estimation, including least squares and truncated series approximations. Accurate and efficient estimates for bit error rates are then proposed.

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EXECUTIVE SUMMARY

DSTO is a sponsoring partner in the Hypatia Scholarship scheme for mathematically talented women, and as such, through Electronic Warfare and Radar Division's Microwave Radar Branch, sponsored the second author of this report to participate in a short research project through the Summer Vacation Scholarship Program. Named after the famous female mathematician Hypatia, the scheme provides both financial and mentoring assets to encourage women to pursue their interests in the mathematical sciences. The work presented here is a report on this project, jointly undertaken by Graham V. Weinberg and Hypatia Scholar Sharon Lee, over the 2006/2007 Summer Period.

This project involves estimation of the Marcum Q-Function, which is an important tool in digital signal processing. It is of interest to both the radar and communications research communities, and has been investigated by the first author quite extensively. Here we examine bit error rate estimation in digital communications, which is intimately related to this function. We show that a method applied in a recent publication, which uses bounds to estimate bit error rates, can be improved considerably by using more direct techniques of estimation.

This work is relevant to the long range research efforts into radar detection issues associated with Task AIR 04/206, EWRD Support for AP-3C E/LM2022 Radar System. Although focusing on a communications application, the results transfer directly to the latter. The technique examined here will be useful for engineers and scientists looking for efficient and accurate approximations for intractable integrals.

Authors

Graham V. Weinberg
Electronic Warfare and Radar Division

Graham V. Weinberg, Ph.D. is a specialist in mathematical analysis and applied probability, and is a graduate of The University of Melbourne. His major DSTO research interest has been the mathematical and stochastic approximation and estimation of radar performance measures.

Sharon Lee
DSTO Summer Vacation Scholarship Program 2006/2007

Sharon Lee is a recipient of the 2005 Hypatia Scholarship for mathematically talented women, studying a Mathematical and Computer Modelling degree at the University of South Australia.

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1 Introduction

This report examines the estimation of bit error rates (BERs) in digital communications [1]. Specifically, we will investigate the recent work of [6] on using bounds to construct approximations for differential quaternary phase shift keying (DQPSK) transmission with Gray coding over an additive white Gaussian noise channel (AWGNC) [14]. In [6], an estimate of this BER is constructed by averaging a lower and upper bound. We show that more direct methods can be applied to estimate the BER, and in some cases more accurate results are obtained.

The BER is a fundamental performance measure of a system, quantifying the reliability or integrity of a received signal [1]. The instantaneous BER, for many practical communication systems, in particular, wireless communications systems, can be written as a function involving the standard Marcum Q-Function ([11], [12] and [13]). This famous function has received much attention in the digital signal processing literature, due to its intractability. Hence, many estimation schemes have been proposed, employing techniques such as Adaptive Simpson Quadrature (ASQ) [10], Taylor Series approximations [16], the construction of lower and upper bounds ([4] – [9] and [19]) and the Monte Carlo scheme ([23] and [24]).

1.1 Bit Error Rates and Marcum's Q-Function

We specialise our analysis to the case considered in [6], as described above. In this scenario, the instantaneous BER is described by

$$\mathrm{BER}(\gamma|a,b) := \mathrm{Q}(a\sqrt{\gamma}, b\sqrt{\gamma}) - \frac{1}{2}e^{-\frac{1}{2}(a^2+b^2)\gamma} I_0(ab\gamma), \tag{1}$$

where constants a and b depend on the modulation/demodulation format, and γ is the total instantaneous signal to noise ratio per bit, and $\mathrm{Q}(\alpha,\beta)$ is the standard Marcum Q-Function, defined by

$$\mathrm{Q}(\alpha,\beta) := \int_{\beta}^{\infty} x e^{-\frac{1}{2}(x^2+\alpha^2)} I_0(\alpha x)dx, \tag{2}$$

and $I_0(\cdot)$ is the modified Bessel function of order zero. In the case considered in [6], $a = \sqrt{2-\sqrt{2}}$ and $b = \sqrt{2+\sqrt{2}}$, so that $b > a$. The key to estimating (1) is to construct good approximations of (2).

Using lower and upper bounds on (2), derived in [5], one can show that

$$\mathrm{BER}(\gamma|a,b) \approx \sqrt{\frac{\pi}{8}} \frac{I_0(ab\gamma)}{\exp(ab\gamma)} \sqrt{\gamma}(a+b)\mathrm{Erfc}\left(\sqrt{\frac{\gamma}{2}}(b-a)\right), \tag{3}$$

where Erfc is the complementary error function (see [6] for details).

Using asymptotic approximations, [6] also proposed the following approximate expression, for large values of γ:

$$\begin{aligned} \mathrm{BER}(\gamma|a,b) &\approx \frac{1}{\sqrt{8\pi ab}}\frac{b+a}{b-a}e^{-\left[-\frac{(b-a)^2}{2}\right]} \\ &= \frac{\sqrt{2}+1}{\sqrt{8\pi\sqrt{2}}}\frac{1}{\sqrt{\gamma}}e^{\left[-(2-\sqrt{2})\gamma\right]}. \end{aligned} \tag{4}$$

Although (3) and (4) produce good results, we will investigate a more direct approach to the estimation of the BER. The Least Squares (LS) approach is an interpolation technique, which may be applied usefully to facilitate the estimation of (2). Additionally, we will consider Taylor Series approximations applied directly to (2). Before considering estimators, we introduce a number of useful representations of the Marcum Q-Function.

1.2 Integral Representations of Marcum's Q-Function

A number of new representations of (2) have been derived in [22]. In the spirit of [4], these convert the Marcum integral to one on a finite domain, with penalty terms added. Converting the Marcum integral (2) to one on a finite domain has the potential to improve the estimation process, since it is somewhat easier to estimate an integral on a finite domain.

It can be shown that

$$\mathrm{Q}(a,b) = \frac{1}{2}\left[1 + e^{-a^2}I_0(a^2)\right] + \int_b^a xe^{-\frac{1}{2}\left(x^2+a^2\right)}I_0(ax)dx. \tag{5}$$

Additionally, by an application of the symmetry relation [15] of the Marcum Q-Function to (5), one can derive

$$\begin{aligned} \mathrm{Q}(a,b) &= \frac{1}{2}\left[1 - e^{-b^2}I_0(b^2)\right] + e^{-\frac{1}{2}\left(a^2+b^2\right)}I_0(ab) \\ &\quad - \int_a^b xe^{-\frac{1}{2}\left(x^2+b^2\right)}I_0(bx)dx. \end{aligned} \tag{6}$$

Further details of the derivation of (5) and (6) can be found in [22].

It is, of course, not difficult to reduce (2) to an integral over a finite domain. Clearly we can write

$$\mathrm{Q}(a,b) = 1 - \int_0^b xe^{-\frac{1}{2}\left(x^2+a^2\right)}I_0(ax)dx. \tag{7}$$

Also, one can apply the Marcum symmetry relation [15] to (7) to produce

$$\mathrm{Q}(a,b) = e^{-\frac{1}{2}\left(a^2+b^2\right)}I_0(ab) - \int_0^a xe^{-\frac{1}{2}\left(x^2+b^2\right)}I_0(bx)dx. \tag{8}$$

We will construct a number of estimators based upon (5), (6) and (8), using the Least Squares Method.

1.3 Least Squares Method

The Least Squares Method (LSM) [3] fits a smooth curve, with minimum error, to a given set of data points. Error, in this case, refers to the sum of the squares of the errors (SSE) or the residuals of the points from the curve. To fit a polynomial of degree n, $P_n(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0$, to a set of m data points, where $m \geq n+1$, the coefficients of the polynomial are determined such that the SSE is minimised, where

$$SSE = \sum_{k=0}^{m} \left[P_n(x_k) - y_k\right]^2, \tag{9}$$

and (x_k, y_k), $k = 1, \ldots, m$, are the data points.

The minimum SSE is obtained by taking the partial derivatives of (9) and equating them to zero. The resultant set of equations are known as the normal equations [3]. The coefficients are obtained by solving the normal equations.

$$\begin{aligned}
\alpha_0 n + \alpha_1 \sum_{k=0}^{m} x_k + \cdots + \alpha_n \sum_{k=0}^{m} x_k^n &= \sum_{k=0}^{m} y_k \\
\alpha_0 \sum_{k=0}^{m} x_k + \alpha_1 \sum_{k=0}^{m} x_k^2 + \cdots + \alpha_n \sum_{k=0}^{m} x_k^{n+1} &= \sum_{k=0}^{m} x_k y_k \\
&\cdots \\
\alpha_0 \sum_{k=0}^{m} x_k^n + \alpha_1 \sum_{k=0}^{m} x_k^{n+1} + \cdots + \alpha_n \sum_{k=0}^{m} x_k^{2n} &= \sum_{k=0}^{m} x_k^n y_k.
\end{aligned} \tag{10}$$

1.4 Contributions of this Report

This report explores the idea of applying LS approximations to evaluate the BER Function (1), achieved by estimating the Marcum Q-Function (2). Instead of applying bounds to estimate the Marcum Q-Function, as in [6], we show that more direct methods can be applied to produce estimators. Specifically, six estimators can be constructed from the finite interval representations of the Marcum Q-Function. In particular, three of the estimators use an exponential function with a quadratic polynomial power to estimate the modified Bessel function. The second group of three fit a polynomial of degree 5 to the entire integrand. These six estimators are compared to the results obtained from ASQ. Speed and accuracy performance are also analysed. We attempt to identify an optimal LS estimator.

In addition, we examine a Taylor Series approximation of the BER function. Its performance is also compared with ASQ and the best performing LS estimator.

2 Numerical Approximation Schemes

A number of estimators of the BER, based on the LS scheme and Taylor Series approach, are now introduced. The LS estimators are constructed by substituting functional approximations to the integrands in (2), (5), (6) and (8). In some cases, the modified Bessel function is approximated by an exponential function, while in others, we apply a polynomial approximation to the entire integrand. We also introduce a single estimator based upon a truncated Taylor Series. Throughout we assume, as in [6], that $a = \sqrt{2-\sqrt{2}}$ and $b = \sqrt{2+\sqrt{2}}$, and hence $b > a$.

The key problem in terms of estimation of the Marcum Q-Function is the presence of the modified Bessel function in the integrand. Hence, we apply functional approximations to eliminate it from the integrand, to facilitate integration. Many of the proposed functional approximations suggested below have been produced by considering suitable fits to the Bessel function by an appropriate polynomial. This will be done by using Least Squares fits.

2.1 Least Squares on a Finite Interval

To begin, we consider approximating the modified Bessel Function with an exponential function of the form $f(x) = e^{\alpha x^2+\beta x+\gamma}$ on a specified finite interval. This is achieved by fitting a quadratic function, $\alpha x^2 + \beta x + \gamma$, to $\log(I_0(x))$, where the coefficients α, β and γ are real constants. Applying the LS scheme, the coefficients can be determined using the method outlined in Section 1.3. This scheme requires sequential estimation of the values of the modified Bessel Function. However, this is not viewed as a shortcoming because the tendency in the literature is to construct approximations of the integral (2) in terms of such functions anyway.

2.1.1 An Estimator Based on Integral Representation (6): $\widehat{E_1}$

In view of (6), we can fit an exponential function to the modified Bessel function component of the integral. Note that

$$\begin{aligned}
&\int_a^b x\ e^{\frac{1}{2}\left(x^2+b^2\right)} e^{\alpha x^2+\beta x+\gamma} dx \\
&= e^{-\frac{1}{2}b^2+\gamma+\frac{\beta^2}{2-4\alpha}} \int_a^b x e^{-\left(\frac{1}{2}-\alpha\right)\left(x-\frac{\beta}{1-2\alpha}\right)^2} dx \\
&= \frac{\delta}{1-2\alpha}\left[e^{-\left(\frac{1}{2}-\alpha\right)\left(a-\frac{\beta}{1-2\alpha}\right)^2} - e^{-\left(\frac{1}{2}-\alpha\right)\left(b-\frac{\beta}{1-2\alpha}\right)^2}\right] \\
&\quad + \frac{\delta\beta\sqrt{\pi}}{(2-4\alpha)\sqrt{\frac{1}{2}-\alpha}}\left[\mathrm{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(b-\frac{\beta}{1-2\alpha}\right)\right)\right.
\end{aligned}$$

$$- \operatorname{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(a-\frac{\beta}{1-2\alpha}\right)\right)\Bigg], \tag{11}$$

where $\delta = e^{-\frac{1}{2}b^2+\gamma+\frac{\beta^2}{2-4\alpha}}$ and $\operatorname{Erf}(\cdot)$ is the error function

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{12}$$

Hence, applying (11) to (6), and then applying the result to (1), we arrive at the BER estimator:

$$\begin{aligned}
\widehat{E_1} \;=\; & \frac{1}{2}\left[1-e^{-b^2}I_0(b^2)\right]+\frac{1}{2}e^{-\frac{1}{2}\left(a^2+b^2\right)}I_0(ab) \\
& -\frac{\delta}{1-2\alpha}\left[e^{-\left(\frac{1}{2}-\alpha\right)\left(a-\frac{\beta}{1-2\alpha}\right)^2}-e^{-\left(\frac{1}{2}-\alpha\right)\left(b-\frac{\beta}{1-2\alpha}\right)^2}\right] \\
& +\frac{\delta\beta\sqrt{\pi}}{(2-4\alpha)\sqrt{\frac{1}{2}-\alpha}}\left[\operatorname{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(b-\frac{\beta}{1-2\alpha}\right)\right)\right. \\
& \left.-\operatorname{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(a-\frac{\beta}{1-2\alpha}\right)\right)\right],
\end{aligned} \tag{13}$$

where the coefficients of the polynomial, α, β and γ, are evaluated numerically using the LSM discussed in Section 1.3. This is easily done in Matlab by specifying it to fit a quadratic expression to the logarithm of data points generated from the modified Bessel function of order zero.

2.1.2 An Approximation Based on Integral Representation (5): $\widehat{E_2}$

Next we consider an approximation to the integrand in (5). We can, as previously, apply the LSM to approximate the modified Bessel Function $\log(I_0(ax))$ with a quadratic curve, where $x \in [a, b]$. As before, note that

$$\begin{aligned}
& \int_a^b x\, e^{-\frac{1}{2}\left(x^2+a^2\right)}e^{\alpha x^2+\beta x+\gamma}dx \\
&= e^{-\frac{1}{2}a^2+\gamma+\frac{\beta^2}{2-4\alpha}}\int_a^b xe^{-\left(\frac{1}{2}-\alpha\right)\left(x-\frac{\beta}{1-2\alpha}\right)^2}dx \\
&= \frac{\delta}{1-2\alpha}\left[e^{-\left(\frac{1}{2}-\alpha\right)\left(a-\frac{\beta}{1-2\alpha}\right)^2}-e^{-\left(\frac{1}{2}-\alpha\right)\left(b-\frac{\beta}{1-2\alpha}\right)^2}\right] \\
&\quad +\frac{\delta\beta\sqrt{\pi}}{(2-4\alpha)\sqrt{\frac{1}{2}-\alpha}}\left[\operatorname{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(b-\frac{\beta}{1-2\alpha}\right)\right)\right. \\
&\quad \left.-\operatorname{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(a-\frac{\beta}{1-2\alpha}\right)\right)\right],
\end{aligned} \tag{14}$$

where $\delta = e^{-\frac{1}{2}a^2+\gamma+\frac{\beta^2}{2-4\alpha}}$. Hence, by applying (14) to (5), and by an application of the result to (1), we arrive at the estimator:

$$\begin{aligned}\widehat{E_2} &= \frac{1}{2}\left[1+e^{-a^2}I_0(a^2)\right] - \frac{\delta}{1-2\alpha}\left[e^{-\left(\frac{1}{2}-\alpha\right)\left(a-\frac{\beta}{1-2\alpha}\right)^2} - e^{-\left(\frac{1}{2}-\alpha\right)\left(b-\frac{\beta}{1-2\alpha}\right)^2}\right] \\ &\quad -\frac{1}{2}e^{-\frac{1}{2}\left(a^2+b^2\right)}I_0(ab) \\ &\quad -\frac{\delta\beta\sqrt{\pi}}{(2-4\alpha)\sqrt{\frac{1}{2}-\alpha}}\left[\mathrm{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(b-\frac{\beta}{1-2\alpha}\right)\right)\right. \\ &\quad \left. -\,\mathrm{Erf}\left(\sqrt{\frac{1}{2}-\alpha}\left(a-\frac{\beta}{1-2\alpha}\right)\right)\right]. \end{aligned} \tag{15}$$

2.2 Least Squares on a Semi-infinite Interval: $\widehat{E_3}$

An LS estimator is now proposed, based upon the original Marcum Q-Function (2). As in the previous Subsections, we can apply a quadratic approximation to the logarithm of the modified Bessel function in the integrand of (2). By doing this, and applying the result to (1), we arrive at the estimator:

$$\begin{aligned}\widehat{E_3} &= \frac{\delta}{1-2\alpha}\left[e^{-\left(\frac{1}{2}\right)-\alpha\left(b-\frac{\beta}{1-2\alpha}\right)^2}\right] \\ &\quad -\frac{1}{2}e^{-\frac{1}{2}\left(a^2+b^2\right)}I_0(ab) \\ &\quad +\frac{\delta\beta\sqrt{\pi}}{(2-4\alpha)\sqrt{\frac{1}{2}-\alpha}}\mathrm{Erfc}\left(\sqrt{\frac{1}{2}-\alpha}\left(b-\frac{\beta}{1-2\alpha}\right)\right), \end{aligned} \tag{16}$$

where $\delta = e^{\frac{1}{2}a^2+\gamma+\frac{\beta^2}{2-4\alpha}}$ and $\mathrm{Erfc}(\cdot)$ is the complementary error function

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt. \tag{17}$$

Next we consider polynomial approximations of the entire integrand, using the LSM.

2.3 Polynomial Integrand Approximations

A number of polynomial approximations are now examined. In particular, we apply an approximation with a polynomial of degree 5 to the entire integrand of the respective integral.

2.3.1 An Estimator Based on Integral Representation (6): $\widehat{E_4}$

To begin, we apply this idea to the integrand of (6). In particular, $xe^{-\frac{1}{2}x^2}I_0(bx)$, can be approximated by a fifth order polynomial, $f(x) = \alpha_1 x^5 + \alpha_2 x^4 + \alpha_3 x^3 + \alpha_4 x^2 + \alpha_5 x + \alpha_6$, where $x \in [a, b]$. It can be shown that

$$e^{-\frac{1}{2}b^2} \int_a^b \left(\alpha_1 x^5 + \alpha_2 x^4 + \alpha_3 x^3 + \alpha_4 x^2 + \alpha_5 x + \alpha_6\right) dx = e^{-\frac{1}{2}b^2}\left[g(b) - g(a)\right], \tag{18}$$

where $g(t) = \frac{\alpha_1}{6}t^6 + \frac{\alpha_2}{5}t^5 + \frac{\alpha_3}{4}t^4 + \frac{\alpha_4}{3}t^3 + \frac{\alpha_5}{2}t^2 + \alpha_6 t$.

Applying (18) to (6), then (1), results in the following estimator:

$$\widehat{E_4} = \frac{1}{2}\left[1 - e^{-b^2}I_0(b^2)\right] + \frac{1}{2}e^{-\frac{1}{2}\left(a^2+b^2\right)}I_0(ab) - e^{-\frac{1}{2}b^2}\left[g(b) - g(a)\right]. \tag{19}$$

The coefficients of the polynomial are estimated by the LSM in Matlab.

2.3.2 An Approximation Based on Integral Representation (8): $\widehat{E_5}$

Using an argument similar to that as in the construction of (19), one can construct the estimator

$$\widehat{E_5} = \frac{1}{2}e^{-\frac{1}{2}\left(a^2+b^2\right)}I_0(ab) + e^{-\frac{1}{2}b^2}g(a) \tag{20}$$

based upon (8).

2.3.3 An Estimator Based on the Transformed Integral Representation: $\widehat{E_6}$

Consider the transformation $a = \sqrt{2\varsigma}$, $b = \sqrt{2\tau}$ and $v = \frac{x^2}{2}$ (see [22] for details of this transformation) of the integral (8). By applying the transformation to (8) and (1), we arrive at the following expression for the BER:

$$\mathrm{BER}(\gamma|\varsigma, \tau) = \frac{1}{2}e^{-(\tau+\varsigma)}I_0(2\sqrt{\tau\varsigma}) + \int_0^{\varsigma} e^{-(v+\tau)}I_0(2\sqrt{\tau v})dv. \tag{21}$$

This suggests one can apply the LSM on the interval $v \in [0, \varsigma]$. In view of this, the final LS estimator we consider is based on the approximation of the integrand, $e^{-(v+\tau)}I_0(2\sqrt{\tau v})$, by a fifth order polynomial using the LSM on the interval $v \in [0, \varsigma]$. This leads to the following estimator:

$$\widehat{E_6} = \frac{1}{2}e^{-(\tau+\varsigma)}I_0(2\sqrt{\tau\varsigma}) + e^{-\tau}g(\varsigma). \tag{22}$$

Next, we consider the Taylor Series Approach.

2.4 Taylor Series Approaches: $\widehat{E_7}$

Based upon Taylor Series, we derive an alternative estimator of the BER function (1) using some known series representations of the Marcum Q-Function and the zeroth order modified Bessel Function of the first kind. A widely known double series expansion ([24] and [16]) of the standard Marcum Q-Function (2) is given by

$$\mathrm{Q}(a,b) = e^{-\left(a^2+b^2\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{a^2}{2}\right)^k}{k!} \sum_{j=0}^{k} \frac{\left(\frac{b^2}{2}\right)^j}{j!}. \tag{23}$$

A well-known series representation of the zeroth order modified Bessel function $I_0(x)$ of the first kind (see [2] and [21]) is

$$I_0(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k} \frac{1}{(k!)^2}. \tag{24}$$

Equations (23) and (24) can be applied to the BER equation in (1). This yields a series representation of the BER function, involving double series:

$$\begin{aligned}
\mathrm{BER}(\gamma|a,b) &= e^{-\left(a^2+b^2\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{a^2}{2}\right)^k}{k!} \sum_{j=0}^{k} \frac{\left(\frac{b^2}{2}\right)^j}{j!} - \frac{1}{2} e^{-\frac{1}{2}\left(a^2+b^2\right)} I_0(ab) \\
&= e^{-\left(a^2+b^2\right)} \sum_{k=0}^{\infty} \left[\frac{\left(\frac{a^2}{2}\right)^k}{k!} \sum_{j=0}^{k} \frac{\left(\frac{b^2}{2}\right)^j}{j!} - \frac{1}{2}\left(\frac{1}{k!}\right)^2 \left(\frac{ab}{2}\right)^{2k} \right] \\
&= e^{-\left(a^2+b^2\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{a^2}{2}\right)^k}{k!} \left[\sum_{j=0}^{k} \frac{\left(\frac{b^2}{2}\right)^j}{j!} - \left(\frac{1}{2}\right)^{k+1} \left(\frac{1}{k!}\right) b^{2k} \right].
\end{aligned} \tag{25}$$

Hence, we can easily construct an estimator of (1) by truncating this series to one with N terms. As N increases without bound, the approximation becomes more accurate. Thus we can propose the following estimator:

$$\widehat{E_7} = e^{-\left(a^2+b^2\right)} \sum_{k=0}^{N} \frac{\left(\frac{a^2}{2}\right)^k}{k!} \left[\sum_{j=0}^{k} \frac{\left(\frac{b^2}{2}\right)^j}{j!} - \left(\frac{1}{2}\right)^{k+1} \left(\frac{1}{k!}\right) b^{2k} \right]. \tag{26}$$

The next section will examine the performance of the six LS estimators and this truncated Taylor Series estimator. The results will be compared to that of ASQ.

3 Performance Analysis

We now examine the performance of the seven estimators introduced in the previous section. ASQ with a tolerance of 10^{-18} will be used as a benchmark of performance. As ASQ is known to be reliable, it has been used by members of the Maritime Air Radar Group in EWRD at DSTO extensively to estimate intractable integrals. It is interesting to see whether any of the seven estimators have the same level of accuracy for less computation time. Throughout we will measure γ in decibels (dB), and will be interested in values of it ranging from 0 to 12dB, as considered in [6]. Appendix A contains all the figures, while Appendix B contains tables, from this analysis.

3.1 LS Estimators $\widehat{E_1}$, $\widehat{E_2}$ and $\widehat{E_3}$

We begin by analysing the first three estimators, $\widehat{E_1}$, $\widehat{E_2}$ and $\widehat{E_3}$. Figure A.1 displays a plot comparing estimates of $\widehat{E_1}$ and ASQ, while Table B.1 displays corresponding numerical estimates. Included in the table are the absolute and relative errors of the estimates, when compared to ASQ. The estimates are obtained by fitting five points to the modified Bessel Function. These results show that $\widehat{E_1}$ performs well on the region $\gamma < 1$, with increasing accuracy for small SNR values.

Next we examine $\widehat{E_2}$. Figure A.2 provides a plot of $\widehat{E_2}$ in comparison to ASQ. Table B.2 provides the approximations of BER using $\widehat{E_2}$. Observe that $\widehat{E_2}$ outperforms $\widehat{E_1}$ for extremely small values of γ, more specifically, for $\gamma \leq 3$. However, the accuracy worsens dramatically for larger values of γ. Beyond $\gamma > 8$, $\widehat{E_2}$ becomes relatively inaccurate, and thus results are not included in Table B.2. The approximation in $\widehat{E_2}$ performs well only on the interval where γ is extremely small.

The performance of $\widehat{E_3}$ can be viewed in Figure A.3, with numerical results in Table B.3. It is interesting to note that the estimators $\widehat{E_1}$ and $\widehat{E_2}$ are generally superior to estimator $\widehat{E_3}$ for small values of γ, but their performance deteriorates quickly. In contrast, $\widehat{E_3}$ provides better approximations for larger values of γ, and the accuracy improves as γ increases. However, these three estimators have relatively large errors and are not entirely consistent with ASQ. The accuracy can be slightly improved by increasing the number of points being fitted to the modified Bessel Function.

As a final comparison of these three estimators, Figure A.4 shows all on the same plot, with ASQ.

3.2 LS Estimators $\widehat{E_4}$, $\widehat{E_5}$ and $\widehat{E_6}$

We now examine the estimators $\widehat{E_4}$, $\widehat{E_5}$ and $\widehat{E_6}$. Figure A.5 provides a plot comparing $\widehat{E_4}$ and ASQ. In Table B.4, estimates of BER using $\widehat{E_4}$ are compared with those obtained via ASQ. The actual error and relative error are given, as previously. One hundred points equally spaced on the interval $[a, b]$ are used for the LS computation. As can be observed, this estimator gives better results than the estimators $\widehat{E_1}$, $\widehat{E_2}$ and $\widehat{E_3}$ for small values of γ, with increasing accuracy for smaller SNR values. However, the estimator's performance deteriorates rapidly for larger values of γ. To improve the accuracy, one can increase the order of the polynomial and the number of points being fit to the modified Bessel Function.

Figure A.6 is a plot of the performance of estimator $\widehat{E_5}$. As can be seen, there is a uniform improvement in performance. Table B.5 confirms the improvement in terms of errors. We observe that $\widehat{E_5}$ is generally superior to $\widehat{E_4}$. The results are accurate for a larger range of γ values. Although the accuracy of the approximation decreases as γ increases, the rate of deterioration is relatively slower than all estimators considered thus far.

We now consider the final LS estimator, $\widehat{E_6}$. Figure A.7 shows its performance relative to ASQ. This estimator proved to be the most accurate and efficient LS estimator. Included in Figure A.7 are estimates derived from the results in [6], based upon (3). Table B.6 contains results using $\widehat{E_6}$, (3) and ASQ. Estimator $\widehat{E_6}$ used 50 points for fitting the modified Bessel function. Relative errors are given, with ϵ_1 that between $\widehat{E_6}$ and ASQ, and ϵ_2 the relative error given in [6]. As shown in Table B.6, $\widehat{E_6}$ outperforms (3) and other LS estimators discussed so far. It is worth noting that $\widehat{E_6}$ provides extremely accurate results for the entire range of γ values of interest. Notice that the actual error is relatively consistent. As for γ larger than $12dB$, the proposed approximation (3) in [6] remains a better choice. $\widehat{E_6}$ may be of practical interest when γ is less than $12dB$, which as pointed out in [6], is the region of interest.

Figure A.8 shows the three estimators considered in this subsection in the same plot, with ASQ.

In addition, we experiment with the degree of the approximating polynomial, n, in $\widehat{E_6}$. For comparison, we include the results of $\widehat{E_6}$ using $n = 6$ with (3) and (4) in [6] on the interval γ between $0dB$ and $14dB$ in Table B.7 . The corresponding relative errors are shown. Note that $\widehat{E_6}$ using $n = 6$ provides better results, with accuracy as high as 2×10^{-17} for some values of γ. The approximations are of extremely high accuracy for the region of interest. Figure A.9 provides a corresponding comparison plot. Figure A.10 shows an enlarged image of Figure A.9 around $\gamma = 13$. The efficiency performance of $\widehat{E_6}$ is analyzed in the subsequent subsection. It is important to note that $\widehat{E_6}$ is capable of improved accuracy results by increasing the value of n, namely the degree of the LS polynomial. One can vary the value of n to obtain the desired accuracy.

This completes our examination of LS estimators. We now investigate the Taylor Series estimator, $\widehat{E_7}$.

3.3 Taylor Series Estimator $\widehat{E_7}$

The final estimator we consider is $\widehat{E_7}$, which is based upon the truncated series in (26). The partial sum uses 80 terms. Figure A.11 contains estimates of $\widehat{E_7}$. It is worth noting that the estimates given by (26) are extremely accurate, especially for small values of γ. It clearly outperforms the LS estimators given in the previous subsections. We do not include a comparison plot between $\widehat{E_6}$ and $\widehat{E_7}$ because the errors in Table B.8, when compared to the results in Table B.6, indicate an enormous improvement. Note that, for larger values of γ, the relative error increases, but the accuracy can be further improved by increasing N, the number of terms in the partial sum.

We now consider computational times associated with these estimators. We will, in particular, be interested in how well $\widehat{E_6}$ and $\widehat{E_7}$ perform relative to ASQ.

3.4 Time Performance Analysis

The previous subsections identified estimators $\widehat{E_6}$ and $\widehat{E_7}$ as having the most consistent performance with ASQ, and in particular, $\widehat{E_7}$ is the more accurate of the two. It will thus be useful to investigate processing speeds associated with these estimators. Specifically, we will be interested in knowing whether either one has comparable performance to ASQ in shorter processing times.

The time performance of $\widehat{E_7}$ is compared with $\widehat{E_6}$ and ASQ in Figures A.12 and A.13, as well as in Table B.9. Computation time is measured in seconds, while accuracy is specified in negative powers of 10. Note that in view of Table B.9, $\widehat{E_6}$ is the most efficient estimator, while $\widehat{E_7}$ is the second and ASQ the least. Moreover, $\widehat{E_6}$ and $\widehat{E_7}$ are very consistent. At $\gamma = 0dB$, $\widehat{E_6}$ obtained an exact result with a polynomial of degree $n = 8$, while $\widehat{E_7}$ with a Taylor Series with partial sum $N = 12$. In contrast, for ASQ, time complexity increases as the accuracy increases. A plot of the results in Table B.9 is given in Figure A.12. Figure A.13 gives an enlarged view of Figure A.12 at an accuracy of 10^{-15}. Notice that $\widehat{E_6}$ slightly outperforms $\widehat{E_7}$.

4 Conclusion

This report investigated numerical estimates of the BER function, in particular, via LS and Taylor Series approximations on the Marcum Q-Function. Using a number of finite integral representations of the Marcum's Q-Function, several LS estimators were derived. Three were based on approximating the modified Bessel function by an exponential function with quadratic argument $(\widehat{E_1}, \widehat{E_2}, \widehat{E_2})$. Another three were based upon a fifth order polynomial approximation of the entire integrand $(\widehat{E_4}, \widehat{E_5}, \widehat{E_6})$. An estimator, based upon Taylor Series was also introduced $(\widehat{E_7})$.

Their accuracy and efficiency performances were analysed. All LS estimators perform well on certain local regions. Results indicated that the estimators, $\widehat{E_4}$, $\widehat{E_5}$ and $\widehat{E_6}$, were generally superior to $\widehat{E_1}$, $\widehat{E_2}$ and $\widehat{E_3}$. The optimal LS estimator identified in this report is $\widehat{E_6}$, providing the *highest accuracy* on the *entire region* of interest. It is the *most efficient* estimator, with higher efficiency than the Taylor Series estimator $\widehat{E_7}$.

From the accuracy perspective, $\widehat{E_7}$ outperforms the LS estimators derived in this report. High accuracy results were obtained at fast speed. It is worth noting that the series approximation requires *much* less computation time than the ASQ approach to achieve the same accuracy.

The results presented here also demonstrated that the estimates (3) and (4) from [6] can be improved significantly within the region of interest (0 to 12dB), by using $\widehat{E_6}$, but for estimates greater than 12dB, the estimate (4) is suitable.

It is worth noting that the Least Squares Method is already available in the computer language Matlab, and so the general methodology used here may be applied to estimate other intractable integrals of interest.

5 Acknowledgements

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Appendix A: Comparsion Plots of the Estimators

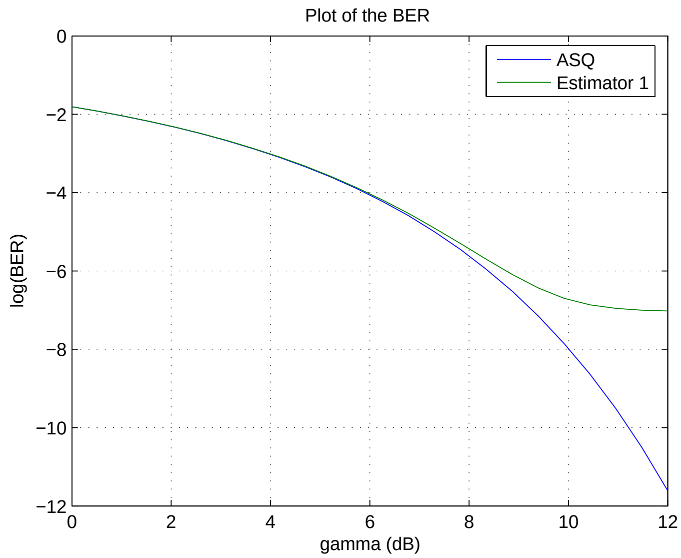

Figure A.1: A plot comparing $\widehat{E_1}$ and ASQ for the region $\gamma = 0dB$ to $\gamma = 12dB$. The BER is given in a logarithmic scale. Observe that estimator $\widehat{E_1}$ performs well for small values of γ, specifically where $\gamma \leq 6dB$.

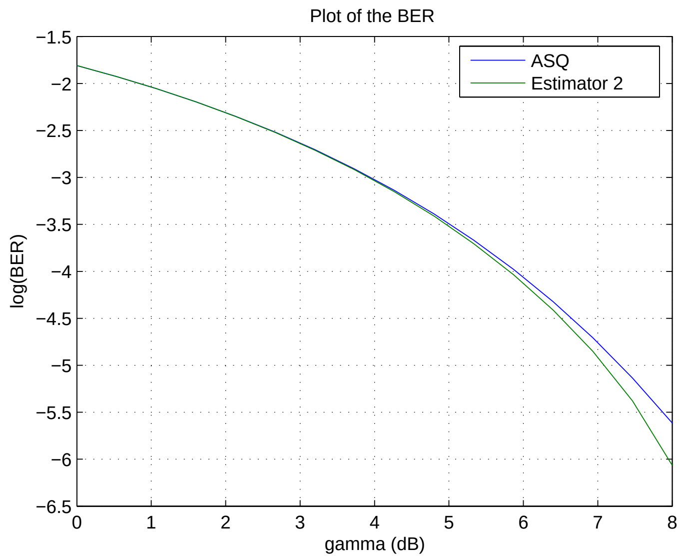

Figure A.2: Performance of $\widehat{E_2}$ in comparision with ASQ on the interval $\gamma = 0dB$ to $\gamma = 8dB$. The BER is given in a logarithmic scale. Estimator $\widehat{E_2}$ provides good approximation for $\gamma \leq 5dB$, with increasing accuracy for smaller SNR values.

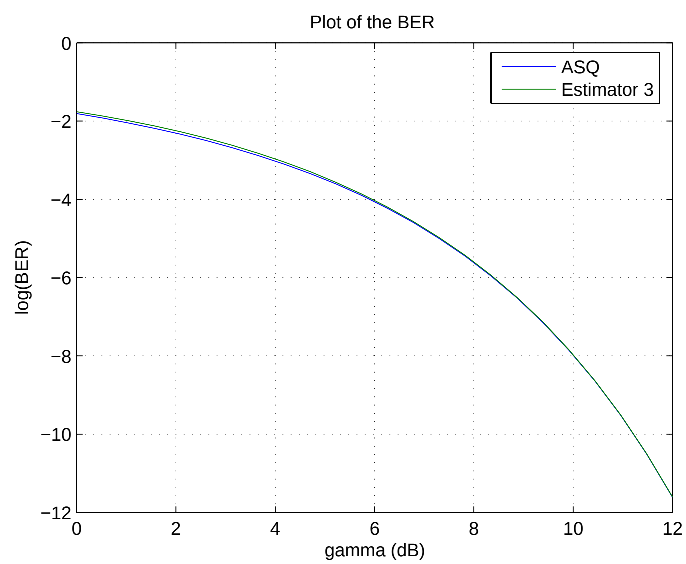

Figure A.3: A plot displaying the performance of $\widehat{E_3}$ in comparision to ASQ for $\gamma = 0dB$ to $\gamma = 12dB$. The BER are given in a logarithmic scale. Estimator $\widehat{E_3}$ performs well for larger values of γ, in particular, $\gamma \geq 10dB$. Observe that accuracy increases as the SNR increases.

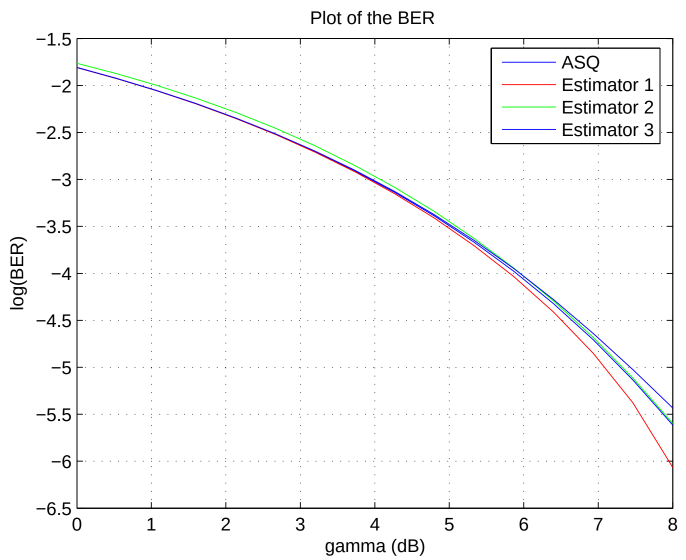

Figure A.4: A Comparison of the three estimators $\widehat{E_1}$, $\widehat{E_2}$ and $\widehat{E_3}$, with ASQ.

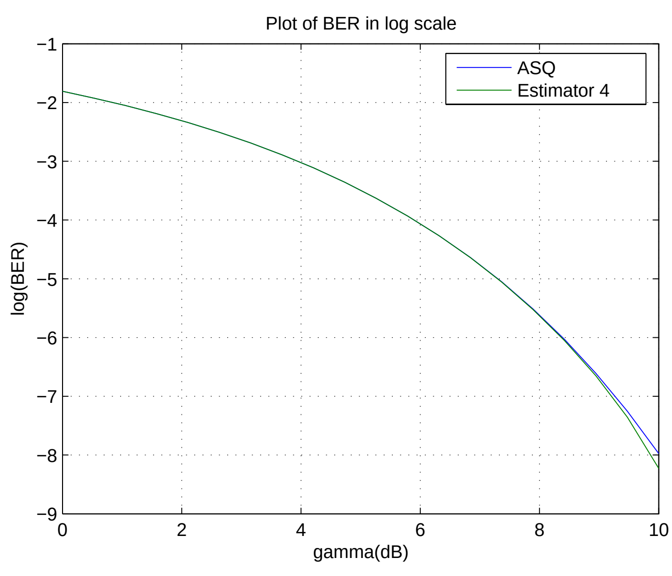

Figure A.5: Performance of $\widehat{E_4}$ in comparision with ASQ on the interval $\gamma = 0dB$ to $\gamma = 10dB$. The BER is given in a logarithmic scale. Notice that for γ beyond $8dB$, $\widehat{E_4}$ begins to deviate from ASQ. Estimator $\widehat{E_4}$ provides good approximation for the region $\gamma \leq 8dB$, with increasing accuracy for smaller SNR values.

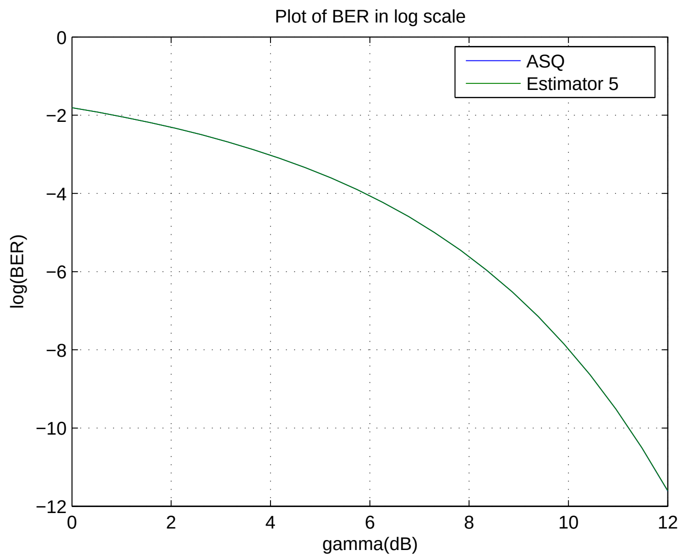

Figure A.6: A plot comparing $\widehat{E_5}$ and ASQ for the region $\gamma = 0dB$ to $\gamma = 12dB$. Observe that $\widehat{E_5}$ covers ASQ. For the range of γ values of interest, approximation $\widehat{E_5}$ provides good results. High accuracy approximations are obtained for $\gamma \leq 9dB$.

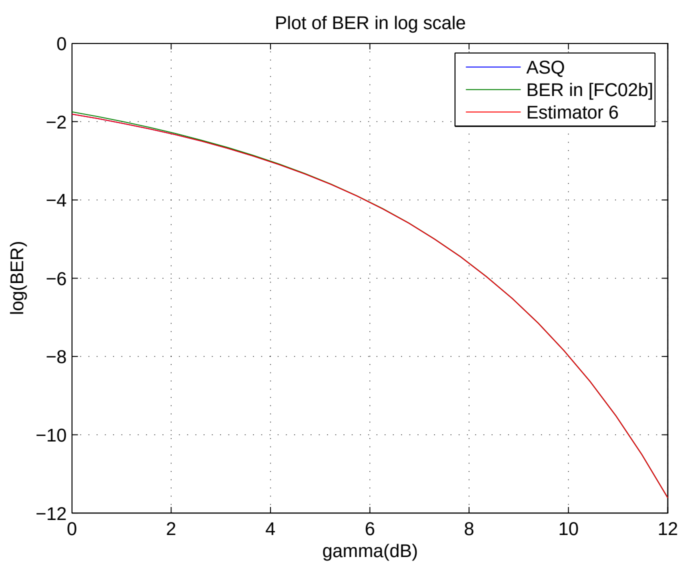

Figure A.7: Performance of $\widehat{E_6}$ in comparision with ASQ on the interval $\gamma = 0dB$ to $\gamma = 12dB$. The BER is given in a logarithmic scale. Approximation $\widehat{E_6}$ performs extremely well on the region of interest, providing high accuracy results for $\gamma \leq 12dB$.

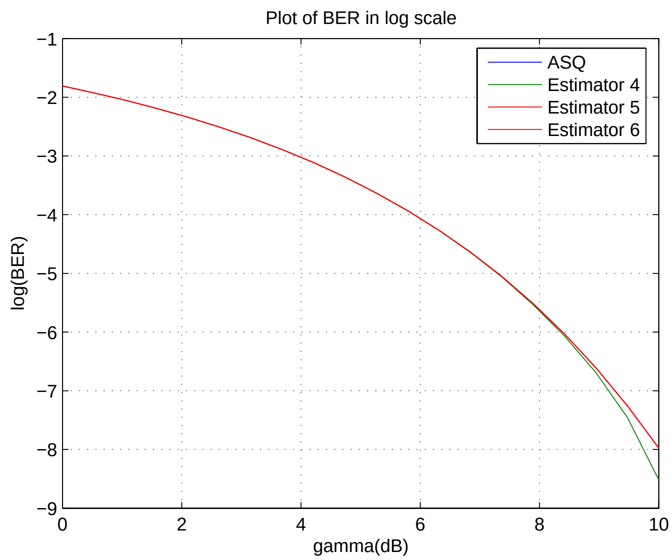

Figure A.8: A Comparison of the three estimators $\widehat{E_4}$, $\widehat{E_5}$ and $\widehat{E_6}$, with ASQ.

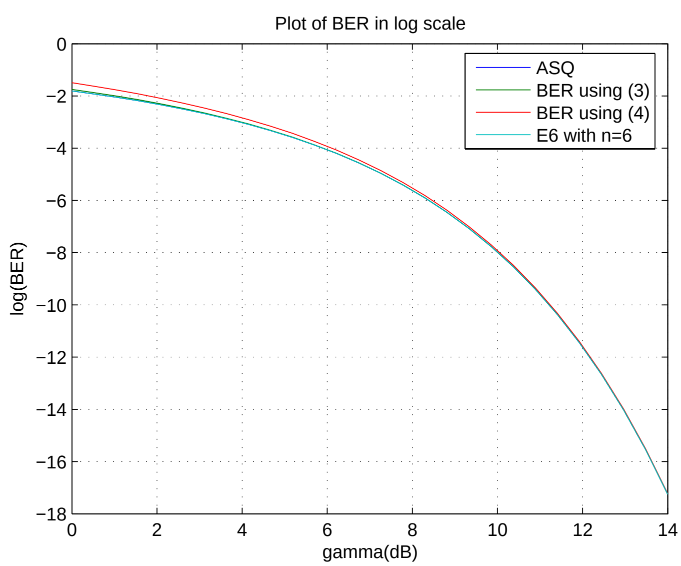

Figure A.9: A comparison plot of $\widehat{E_6}$ using $n = 6$, (3), (4) and ASQ for $\gamma = 0dB$ to $\gamma = 14dB$. The BER is given in a logarithmic scale. Estimator $\widehat{E_6}$ with $n = 6$ performs extremely well on this region, providing extremely high accuracy results. Observe that $\widehat{E_6}$ with $n = 6$ is almost exactly on ASQ.

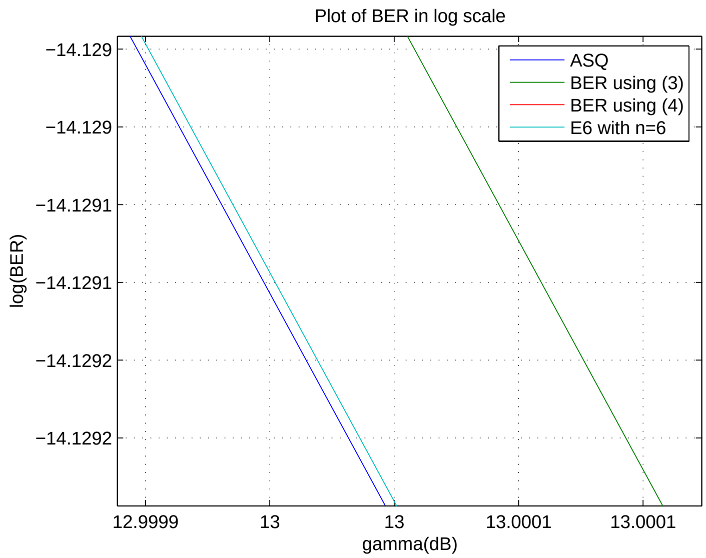

Figure A.10: An enlarged view of Figure A.9 around $\gamma = 13$*. Note that BER using (4) falls outside the region of the Figure. Observe how closely* $\widehat{E_6}$ *approximate ASQ.*

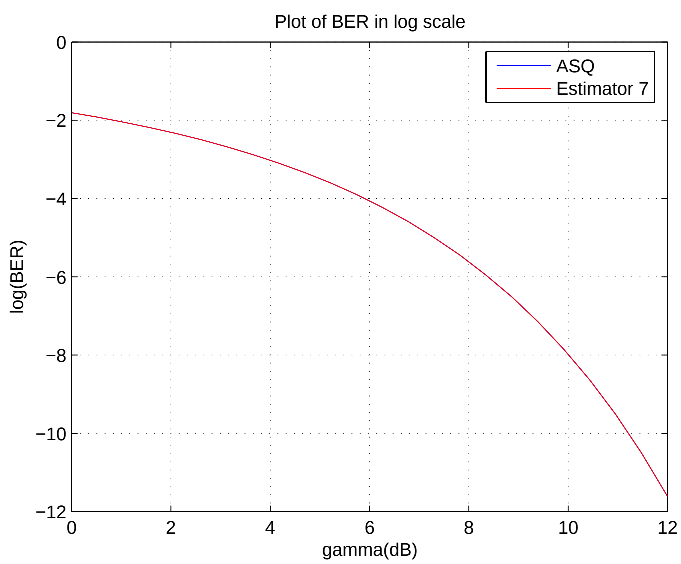

Figure A.11: A plot of $\widehat{E_7}$ *and ASQ for* $\gamma = 0dB$ *to* $\gamma = 12dB$*. The BER is given in a logarithmic scale. Extremely high accuracy results is obtained using the series in (26) for the entire region of interest.*

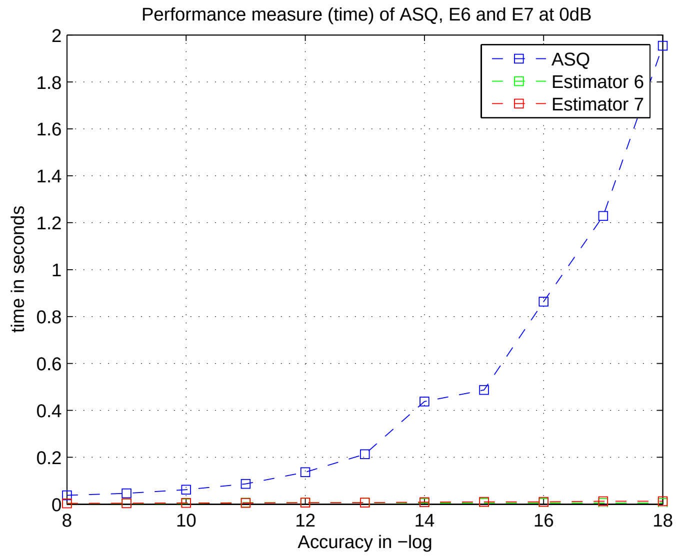

Figure A.12: Time performance plot of $\widehat{E_6}$, $\widehat{E_7}$ and ASQ. Time is in seconds and accuracy is given in a logarithmic scale, base 10. Observe that both $\widehat{E_6}$ and $\widehat{E_7}$ are extremely efficient in comparision with ASQ. Both $\widehat{E_6}$ and $\widehat{E_7}$ are very consistent, while ASQ depends heavily on the degree of accuracy.

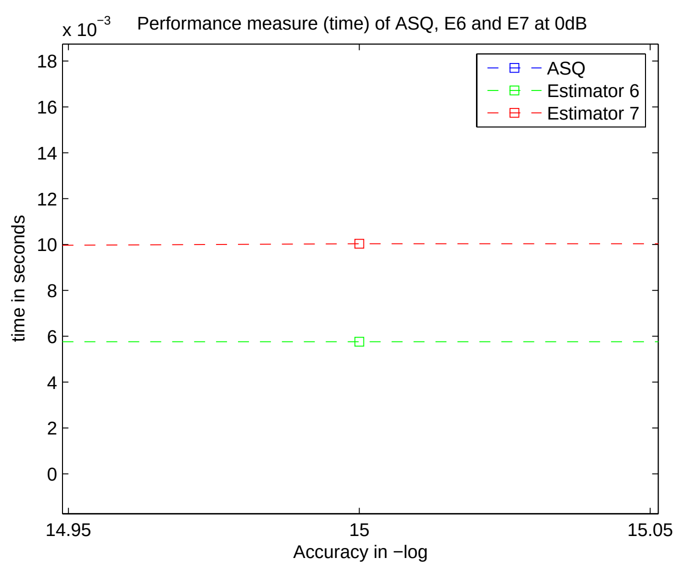

Figure A.13: An enlarged view of Figure A.12 at 10^{-15}, suggesting $\widehat{E_6}$ is more efficient than $\widehat{E_7}$. Note that ASQ falls outside the region of the Figure.

Appendix B: Tables of Numerical Results

$\gamma(dB)$	ASQ	$\widehat{E_1}$	ϵ_1	ϵ_2
0	1.63907530399585e-001	1.64066250992386e-001	1.5872e-004	9.6835e-002%
1	1.30332229277756e-001	1.30554573532828e-001	2.2234e-004	1.7060e-001%
2	9.93324252133400e-002	9.96169239202047e-002	2.8450e-004	2.8641e-001%
3	7.18727612659082e-002	7.22160657416420e-002	3.4330e-004	4.7766e-001%
4	4.87488622380308e-002	4.91535216927732e-002	4.0466e-004	8.3009e-001%
5	3.04943244285625e-002	3.09694472029306e-002	4.7512e-004	1.5581e+000%
6	1.72359006046928e-002	1.77913923593411e-002	5.5549e-004	3.2229e+000%
7	8.58004434130124e-003	9.22092893335622e-003	6.4088e-004	7.4695e+000%
8	3.64294312896472e-003	4.36665001049495e-003	7.2371e-004	1.9866e+001%
9	1.26710356868390e-003	2.06276280461294e-003	7.9566e-004	6.2794e+001%
10	1.19230674177357e-003	3.43184596033452e-004	8.4912e-004	2.4742e+002%
11	9.46488998159275e-004	9.46488998159275e-004	8.7859e-004	1.2940e+003%
12	9.05258912217385e-006	8.90723374721225e-004	8.8167e-004	9.7394e+003%

Table B.1: A comparision of $\widehat{E_1}$ and ASQ approximations, with a tolerance of 10^{-18}. Five equally spaced points are used for fitting the modified Bessel function. ϵ_1 represents the absolute error between $\widehat{E_1}$ and ASQ, and ϵ_2 is the relative error

$\gamma(dB)$	ASQ	$\widehat{E_2}$	ϵ_1	ϵ_2
0	1.63907530399585e-001	1.63896182918470e-001	1.1347e-005	6.9231e-003%
1	1.30332229277756e-001	1.30283201772424e-001	4.9028e-005	3.7617e-002%
2	9.93324252133400e-002	9.91910467676262e-002	1.4138e-004	1.4233e-001%
3	7.18727612659082e-002	7.15574732746942e-002	3.1529e-004	4.3868e-001%
4	4.87488622380308e-002	4.81818860883209e-002	5.6698e-004	1.1631e+000%
5	3.04943244285625e-002	2.96511574388189e-002	8.4317e-004	2.7650e+000%
6	1.72359006046928e-002	1.61643524537770e-002	1.0715e-003	6.2170e+000%
7	8.58004434130124e-003	7.35773400350712e-003	1.2223e-003	1.4246e+001%
8	3.64294312896472e-003	2.31931668012224e-003	1.3236e-003	3.6334e+001%

Table B.2: Approximations of $BER(\gamma|a,b)$ based on estimator $\widehat{E_2}$ on the interval $\gamma = 0dB$ to $\gamma = 8dB$. Five equally spaced points are used for fitting the modified Bessel function. $\epsilon_1 = |ASQ - \widehat{E_2}|$ and $\epsilon_2 = 100 \times \frac{\epsilon_1}{ASQ}$.

$\gamma(dB)$	ASQ	$\widehat{E_3}$	ϵ_1	ϵ_2
0	1.63907530399585e-001	1.71550949650647e-001	7.6434e-003	4.6633e+000%
1	1.30332229277756e-001	1.37826804088733e-001	7.4946e-003	5.7504e+000%
2	9.93324252133400e-002	1.05719011266452e-001	6.3866e-003	6.4295e+000%
3	7.18727612659082e-002	7.64823916401551e-002	4.6096e-003	6.4136e+000%
4	4.87488622380308e-002	5.15201561338994e-002	2.7713e-003	5.6848e+000%
5	3.04943244285625e-002	3.18780295476395e-002	1.3837e-003	4.5376e+000%
6	1.72359006046928e-002	1.78165119904009e-002	5.8061e-004	3.3686e+000%
7	8.58004434130124e-003	8.78891826423036e-003	2.0887e-004	2.4344e+000%
8	3.64294312896472e-003	3.70801023832913e-003	6.5067e-005	1.7861e+000%
9	1.26710356868390e-003	1.28431703460022e-003	1.7213e-005	1.3585e+000%
10	1.19230674177357e-003	3.46862265107066e-004	3.6777e-006	1.0716e+000%
11	9.46488998159275e-004	6.84867934099513e-005	5.9167e-007	8.7144e-001%
12	9.05258912217385e-006	9.11841944014908e-006	6.5830e-008	7.2720e-001%

Table B.3: A comparision of $\widehat{E_3}$ with ASQ approximations. Five equally spaced points are used for fitting the modified Bessel function. ϵ_1 represents the relative error of $\widehat{E_3}$ and ϵ_2 is the relative error.

$\gamma(dB)$	ASQ	$\widehat{E_4}$	ϵ_1	ϵ_2
0	1.63907530399585e-001	1.63907488870700e-001	4.1529e-008	2.5337e-005%
1	1.30332229277756e-001	1.30332106347130e-001	1.2293e-007	9.4321e-005%
2	9.93324252133400e-002	9.93320691796630e-002	3.5603e-007	3.5843e-004%
3	7.18727612659082e-002	7.18718252819206e-002	9.3598e-007	1.3023e-003%
4	4.87488622380308e-002	4.87466026625585e-002	2.2596e-006	4.6351e-003%
5	3.04943244285625e-002	3.04892302562418e-002	5.0942e-006	1.6705e-002%
6	1.72359006046928e-002	1.72251622791136e-002	1.0738e-005	6.2302e-002%
7	8.58004434130124e-003	8.55909104642799e-003	2.0953e-005	2.4421e-001%
8	3.64294312896472e-003	3.60577177556770e-003	3.7171e-005	1.0204e+000%
9	1.26710356868390e-003	1.20888066573677e-003	5.8223e-005	4.5950e+000%
10	3.43184596033453e-004	2.67003180762893e-004	7.6181e-005	2.2198e+001%
11	6.78951281813439e-005	-3.69695411045479e-006	7.1592e-005	1.0545e+002%
12	9.05258912217385e-006	-2.75618525530374e-006	1.1809e-005	1.3045e+002%

Table B.4: Estimates of $BER(\gamma|a,b)$ using $\widehat{E_4}$ on the interval $\gamma = 0dB$ to $\gamma = 12dB$. 100 equally spaced points are used for fitting the intergand in (6). $\epsilon_1 = |ASQ - \widehat{E_4}|$ and $\epsilon_2 = 100 \times \frac{\epsilon_1}{ASQ}$.

$\gamma(dB)$	ASQ	$\widehat{E_5}$	ϵ_1	ϵ_2
0	1.63907530399585e-001	1.63907546749063e-001	1.5105e-008	9.2154e-006%
1	1.30332229277756e-001	1.30332244210175e-001	1.3768e-008	1.0564e-005%
2	9.93324252133400e-002	9.93324155663120e-002	1.0629e-008	1.0700e-005%
3	7.18727612659082e-002	7.18726865834308e-002	7.5848e-008	1.0553e-004%
4	4.87488622380308e-002	4.87486871417470e-002	1.7599e-007	3.6102e-004%
5	3.04943244285625e-002	3.04940693189189e-002	2.5575e-007	8.3867e-004%
6	1.72359006046928e-002	1.72356861246986e-002	2.1509e-007	1.2479e-003%
7	8.58004434130124e-003	8.58005712697456e-003	7.3343e-009	8.5480e-005%
8	3.64294312896472e-003	3.64328283674178e-003	3.3717e-007	9.2554e-003%
9	1.26710356868390e-003	1.26763628874464e-003	5.3069e-007	4.1882e-002%
10	3.43184596033453e-004	3.43637039076888e-004	4.4649e-007	1.3010e-001%
11	6.78951281813439e-005	6.81247127723810e-005	2.2577e-007	3.3250e-001%
12	9.05258912217385e-006	9.12061302541548e-006	6.7384e-008	7.4431e-001%

Table B.5: A comparision of the approximation of $BER(\gamma|a,b)$ based on equation (20) with that obtained via ASQ. 50 equally spaced points are used for fitting the intergrand in (8). $\epsilon_1 = |ASQ - \widehat{E_5}|$ and $\epsilon_2 = 100 \times \frac{\epsilon_1}{ASQ}$.

$\gamma(dB)$	ASQ	$\widehat{E_6}$	ϵ_1	BER in [6]	ϵ_2
0	1.63907530399585e-001	1.63907530400071e-001	2.9656e-010%	1.73998678128697e-001	6.1566e+000%
1	1.30332229277756e-001	1.30332229279850e-001	1.6061e-009%	1.36604369075718e-001	4.8124e+000%
2	9.93324252133400e-002	9.93324252171851e-002	3.8709e-009%	1.02741930798427e-001	3.4324e+000%
3	7.18727612659082e-002	7.18727612659082e-002	1.2320e-008%	7.35176792703302e-002	2.2887e+000%
4	4.87488622380308e-002	4.87488621670844e-002	1.4553e-007%	4.94684302616405e-002	1.4761e+000%
5	3.04943244285625e-002	3.04943243033633e-002	4.1057e-007%	3.07841984642038e-002	9.5058e-001%
6	1.72359006046928e-002	1.72359008025161e-002	1.1477e-006%	1.73427752950751e-002	6.2007e-001%
7	8.58004434130124e-003	8.58004526916485e-003	1.0814e-005%	8.61510874905393e-003	4.0867e-001%
8	3.64294312896472e-003	3.64294348167390e-003	9.6820e-006%	3.65278932995171e-003	2.7028e-001%
9	1.26710356868390e-003	1.26710160675231e-003	1.5484e-004%	1.26936632743277e-003	1.7858e-001%
10	3.43184596033453e-004	3.43182757950219e-004	5.3560e-004%	3.43588251021892e-004	1.1762e-001%
11	6.78951281813439e-005	6.78960539104487e-005	1.3635e-003%	6.79475127646939e-005	7.7155e-002%
12	9.05258912217385e-006	9.05419050617861e-006	1.7690e-002%	9.05715038668292e-006	5.0386e-002%

Table B.6: Performance of $\widehat{E_6}$, (3) and ASQ. ϵ_1 represents the relative error of $\widehat{E_6}$ and ϵ_2 is the relative error given in [5]. Observe that $\widehat{E_6}$ outperforms (3).

$\gamma(dB)$	ASQ	$\widehat{E_6}$ with $n = 6$	ϵ_1	BER using (3)	ϵ_2	BER using (4)	ϵ_3
0	1.639075303e-001	1.639075303e-001	1.6934e-014%	1.739986781e-001	6.1566e+000%	2.254210348e-001	3.7529e+001%
1	1.303322292e-001	1.303322292e-001	1.4907e-013%	1.366043690e-001	4.8124e+000%	1.726326057e-001	3.2456e+001%
2	9.933242521e-002	9.933242521e-002	8.6621e-013%	1.027419307e-001	3.4324e+000%	1.271145833e-001	2.7969e+001%
3	7.187276126e-002	7.187276126e-002	1.1334e-011%	7.351767927e-002	2.2887e+000%	8.908322004e-002	2.3946e+001%
4	4.874886223e-002	4.874886223e-002	1.6426e-011%	4.946843026e-002	1.4761e+000%	5.866306626e-002	2.0337e+001%
5	3.049432442e-002	3.049432442e-002	4.7817e-010%	3.078419846e-002	9.5058e-001%	3.571928426e-002	1.7134e+001%
6	1.723590060e-002	1.723590060e-002	2.1835e-009%	1.734277529e-002	6.2007e-001%	1.970599722e-002	1.4331e+001%
7	8.580044341e-003	8.580044339e-003	1.5631e-008%	8.615108749e-003	4.0867e-001%	9.601935885e-003	1.1910e+001%
8	3.642943128e-003	3.642943125e-003	9.1022e-008%	3.652789329e-003	2.7028e-001%	4.001461897e-003	9.8415e+000%
9	1.267103568e-003	1.267103576e-003	5.8926e-007%	1.269366327e-003	1.7858e-001%	3.431846049e-004	8.0898e+000%
10	3.43184596e-004	3.431846049e-004	2.5988e-006%	3.435882510e-004	1.1762e-001%	7.155565315e-005	6.6184e+000%
11	6.78951116e-005	6.789511160e-005	2.4416e-005%	6.794751276e-005	7.7155e-002%	7.155565315e-005	5.3914e+000%
12	9.05258912e-006	9.052581472e-006	8.4507e-005%	9.057150386e-006	5.0386e-002%	9.448643705e-006	4.3750e+000%
13	7.34977835e-007	7.349868908e-007	1.2321e-003%	7.352186001e-007	3.2758e-002%	7.609826574e-007	3.5382e+000%
14	3.19776721e-008	3.198172372e-008	1.2670e-002%	3.198445315e-008	2.1205e-002%	3.288995544e-008	2.8529e+000%

Table B.7: A comparision of $\widehat{E_6}$ with $n = 6$, approximations using [6] and ASQ on the interval γ between $0dB$ and $14dB$.ϵ_1 represents the relative error of $\widehat{E_6}$ with $n = 6$, ϵ_2 indicates the relative error of (3)and ϵ_3 is the relative error of (4). Notice the high degree of accuracy of $\widehat{E_6}$ for small values of γ.

$\gamma(dB)$	ASQ	$\widehat{E_7}$	ϵ_1	ϵ_2
0	1.63907530399585e-001	1.63907530399585e-001	0.0000e+000	0.0000e+000%
1	1.30332229277756e-001	1.30332229277756e-001	0.0000e+000	0.0000e+000%
2	9.93324252133400e-002	9.93324252133400e-002	0.0000e+000	0.0000e+000%
3	7.18727612659082e-002	7.18727612659082e-002	4.1633e-017	5.7926e-016%
4	4.87488622380308e-002	4.87488622380307e-002	5.5511e-017	1.1387e-015%
5	3.04943244285625e-002	3.04943244285625e-002	1.0408e-017	3.4132e-016%
6	1.72359006046928e-002	1.72359006046928e-002	1.7347e-017	1.0065e-015%
7	8.58004434130124e-003	8.58004434130123e-003	5.2042e-018	6.0654e-016%
8	3.64294312896472e-003	3.64294312896472e-003	4.3368e-019	1.1905e-016%
9	1.26710356868390e-003	1.26710356868390e-003	6.5052e-019	5.1339e-016%
10	3.43184596033453e-004	3.43184596033452e-004	3.7947e-019	1.1057e-015%
11	6.78951281813439e-005	6.78951281813436e-005	2.8460e-019	4.1918e-015%
12	9.05258912217385e-006	9.05258912217359e-006	2.5919e-019	2.8632e-014%

Table B.8: An approximation of $BER(\gamma|a,b)$ based on a partial sum of 80 terms using equation (26).$\epsilon_1 = |ASQ - \widehat{E_7}|$ and $\epsilon_2 = 100 \times \frac{\epsilon_1}{ASQ}$.

Accuracy	ASQ	$\widehat{E_6}$	$\widehat{E_7}$
10^{-8}	3.7900e-002	3.7935e-003	3.6175e-003
10^{-9}	4.6300e-002	3.7935e-003	4.3964e-003
10^{-10}	6.1700e-002	3.7935e-003	5.4035e-003
10^{-11}	8.6800e-002	6.5642e-003	5.4035e-003
10^{-12}	1.3660e-001	6.5642e-003	6.6447e-003
10^{-13}	2.1310e-001	6.5642e-003	6.6447e-003
10^{-14}	4.3790e-001	5.7586e-003	8.7316e-003
10^{-15}	4.8680e-001	5.7586e-003	1.0030e-002
10^{-16}	8.6340e-001	5.7586e-003	1.0030e-002
10^{-17}	1.2287e+000	5.7586e-003	1.2531e-002
10^{-18}	1.9540e+000	5.8133e-003	1.2531e-002

Table B.9: Time performance of $\widehat{E_6}$, $\widehat{E_7}$ and ASQ. Time is given in seconds. Accuracy is given by 10^{-m}, where $m \in \{8, 9, \ldots, 18\}$. Observe that $\widehat{E_6}$ is the most efficient estimator, with $\widehat{E_7}$ following closely behind. ASQ is the least efficient, and time increases as the accuracy increases. Both $\widehat{E_6}$ and $\widehat{E_7}$ are very consistent.

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19. ABSTRACT

This report investigates the estimation of bit error rates in digital communications, motivated by recent work in [6]. In the latter, bounds are used to construct estimates for bit error rates in the case of differentially coherent quadrature phase-keying with Gray coding over an additive white Gaussian noise channel. By analysing Marcum's Q-Function, which is an integral part of bit error rate expressions, we derive more direct methods of estimation, including least squares and truncated series approximations. Accurate and efficient estimates for bit error rates are then proposed.

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